Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational Physics 224 (2007) 1049–1063
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

A contribution to the construction of diffusion fluxes for finite volume and discontinuous Galerkin schemes

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Received 3 April 2006; received in revised form 3 November 2006; accepted 6 November 2006
Available online 18 December 2006

Abstract

In this paper, we consider numerical approximations of diffusion terms for finite volume as well as discontinuous Galerkin schemes. Both classes of numerical schemes are quite successful for advection equations capturing strong gradients or even discontinuities, because they allow their approximate solutions to be discontinuous at the grid cell interfaces. But, this property may lead to inconsistencies with a proper definition of a diffusion flux. Starting with the finite volume formulation, we propose a numerical diffusion flux which is based on the exact solution of the diffusion equation with piecewise polynomial initial data. This flux may also be used by discontinuous Galerkin schemes and gives a physical motivation for the Symmetric Interior Penalty discontinuous Galerkin scheme. The flux proposed leads to a one-step finite volume or discontinuous Galerkin scheme for diffusion, which is arbitrary order accurate simultaneously in space and time. This strategy is extended to define suitable numerical fluxes for nonlinear diffusion problems.

Keywords: Numerical diffusion fluxes; Discontinuous Galerkin schemes; Generalized Riemann problem; Nonlinear unsteady diffusion systems; High order accuracy; Space–time approach

1. Introduction

The class of discontinuous Galerkin (DG) finite element schemes seems to be one of the most interesting candidates to construct high order accurate schemes for solving the compressible Navier–Stokes equations in complex three-dimensional geometries. The DG scheme was originally proposed by Reed and Hill [21] in 1973 for the numerical solution of neutron transport and by Nitsche [20] in 1971 for the approximation of elliptic equations. The application to time dependent nonlinear conservation laws then was starting with the work of Cockburn and Shu [9], for a review of the development of DG methods see [8].

To extend this approach to the compressible Navier–Stokes equations a severe difficulty in the DG approach turned out to be the definition of appropriate numerical fluxes for diffusion terms. In the DG approach

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2006.11.004

approximate solutions are allowed to be discontinuous at the grid cell interfaces. By this, it is possible to establish a consistent and stable approximation of strong gradients and even the shock-capturing property. The numerical flux for the advection part takes into account the discontinuity and uses information of the break up of these discontinuities into different waves. How to do this in a proper way has been shown within the development of the finite volume schemes during the last three decades. The problem for the approximation of the diffusion terms seems to be that the discontinuities at the interfaces have to be properly resolved, too. To take simply the arithmetic mean of the derivatives from the right and the left may be inconsistent, because the arithmetic mean of the derivatives does not take into account a possible jump of the functions. A number of corrections and better definitions of the diffusion fluxes have been proposed which introduce in some way the influence of the jump. A unified formulation and analysis has been given by Arnold et al. in [2] and was continued in [3].

It is interesting to observe that the difficulty approximating diffusion terms already occurs within the finite volume framework. The values at the grid cell interfaces may be discontinuous after the reconstruction step, especially at strong gradients. Let us consider the integral conservation equation

$$u_i^{n+1} = u_i^n + \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \kappa u_x\left(x_{i+\frac{1}{2}}, t\right) \mathrm{d}t - \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \kappa u_x\left(x_{i-\frac{1}{2}}, t\right) \mathrm{d}t, \tag{1.1}$$

for the scalar diffusion equation

$$u_t = \kappa u_{xx}, \quad \text{with } \kappa = \text{constant} > 0. \tag{1.2}$$

The FV scheme is based on this evolution equation for the integral values (1.1) which is obtained from Eq. (1.2) by integration with respect to x and t over the spatial grid cell $\left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ and the time interval $[t_n, t_{n+1}]$. Here, u_i^n denotes the integral value at time t_n.

For advection equations the idea of Godunov [13] to take constant values in each grid cell and to calculate the break-up of the discontinuity was very successful. This idea may also directly be applied to diffusion. A general bounded solution for initial value problems of the diffusion Eq. (1.2) reads as

$$u(x,t) = \frac{1}{2\sqrt{\kappa\pi t}}\int_{-\infty}^{\infty} u_0(\xi)\mathrm{e}^{-\frac{(x-\xi)^2}{4\kappa t}}\mathrm{d}\xi. \tag{1.3}$$

Inserting the piecewise constant initial data

$$u(x,0) = u_0(x) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x > 0, \end{cases} \tag{1.4}$$

and differentiating the solution with respect to x gives at $x = 0$:

$$u_x(0,t) = \frac{u_r - u_l}{2\sqrt{\pi\kappa t}}. \tag{1.5}$$

This we can use to define the numerical flux

$$\int_{t_n}^{t_{n+1}} \kappa u_x\left(x_{i+\frac{1}{2}}, t\right) \mathrm{d}t \approx \kappa\Delta t \frac{u_{i+1}^n - u_i^n}{\sqrt{\pi\kappa\Delta t}}. \tag{1.6}$$

Inserting this flux approximation into (1.1) and resorting yields

$$u_i^{n+1} = u_i^n + \frac{1}{\sqrt{\pi\kappa\Delta t}}\frac{\kappa\Delta t}{\Delta x}(u_{i+1}^n - 2u_i^n + u_{i-1}^n). \tag{1.7}$$

It is obvious that this is no consistent approximation to the diffusion Eq. (1.2) until

$$\frac{1}{\sqrt{\pi\kappa\Delta t}} = \frac{1}{\Delta x} \Rightarrow \frac{\kappa\Delta t}{\Delta x^2} = \frac{1}{\pi}. \tag{1.8}$$

In this case, the method coincides formally with the finite difference scheme using central differences.

These considerations motivate the standard approach of the finite volume community in practical calculations of advection diffusion equations – the use of central finite differences for the diffusion part. The values taken for the differences are not the values at the grid cell interfaces obtained from the reconstruction step, but

rather the approximation of the average values used as point values in the barycenter. This procedure may be considered as an additional smooth reconstruction of values at the grid cell interface to evaluate the diffusion flux. To generate no wiggles at strong gradients this approximation needs a couple of points within the viscous profile. This is no problem, if this corresponds well with the physical viscous profile. For high Reynolds number flow or other problems with a narrow layer this approach may lead to a refinement of the grid to get stability. An approach to avoid this smearing was given by Harabetian in [14] for a general advection diffusion equation. He fitted a profile of a traveling wave from one grid cell center to the adjacent one and used a traveling wave solution to obtain the flux. But this approach becomes rather complicated for the Navier–Stokes equations, see Weekes [26].

In this paper, we propose a diffusion flux for (1.2) that is based on the usual reconstruction at the grid cell interfaces for advection equations and may be considered as a self-consistent treatment of diffusion terms in the FV framework. It gives proper results for diffusion in FV schemes even in the case of discontinuous data generated by a discontinuous nonlinear reconstruction, based on TVD, ENO or WENO interpolation, or for DG schemes. The order in space and time is determined by the order of the reconstruction or the order of the trial function for DG schemes. The numerical diffusion flux is based on the initial value problem for (1.2) not with piecewise constant but with piecewise polynomial data which we call diffusive generalized Riemann problem. The piecewise constant discretization cannot work, as the approximation is first order accurate and has no approximation properties for the spatial derivative beside the case (1.8) being identical to the difference approximation.

The format of this paper is as follows. In Section 2, we use the solution of the diffusive generalized Riemann problem to construct the numerical flux for the linear scalar diffusion equation. It is used for a FV scheme and the experimental order of convergence for orders 1 up to 6 is shown. In Section 3, we combine this numerical flux with the discontinuous Galerkin approach. The scheme proposed is compared for P^1 elements with the symmetric interior penalty method. The extension to nonlinear diffusion systems is presented in Section 4. Conclusions and an outlook addressing the advection diffusion problem are given in Section 5.

2. A finite volume scheme for diffusion

The FV scheme is an approximation of the evolution equation for integral mean values (1.1) and may be written as

$$u_i^{n+1} = u_i^n + \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \kappa w_x\left(x_{i+\frac{1}{2}}, t\right) \mathrm{d}t - \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \kappa w_x\left(x_{i-\frac{1}{2}}, t\right) \mathrm{d}t, \tag{2.9}$$

where we introduced the high order ($\geqslant 2$) reconstruction polynomial w. Higher order reconstructions have been proposed in [15] called essentially non-oscillatory (ENO) reconstruction. Some times later weighted essentially non-oscillatory (WENO) reconstruction was introduced, see [18], and became the standard reconstruction technique. A detailed description of the different approaches is given, e.g. by Shu in [22]. To find a proper definition of the remaining numerical fluxes for w_x we consider in the following the local problem at the grid cell interface and deduce from this a diffusion flux approximation.

2.1. The diffusive generalized Riemann problem

We assume that we have already polynomials in every grid cell which may jump at the grid cell interface and consider the initial value problem for the diffusion Eq. (1.2) with piecewise polynomial initial data:

$$u(x,0) = u_0(x) = \begin{cases} \sum_{j=0}^{k} c_j^l x^j & \text{for } x < 0, \\ \sum_{j=0}^{k} c_j^r x^j & \text{for } x > 0. \end{cases} \tag{2.10}$$

We call this initial value problem the diffusive generalized Riemann problem (dGRP) motivated by the name in the hyperbolic case, which was introduced by Ben-Artzi and Falcovitz [6] for piecewise linear initial data,

constructing a second order accurate finite volume scheme for the Euler equations. General polynomial data were considered by Toro et al. within the ADER approach, see [23,24].

The general bounded solution for initial value problems of the diffusion equation is given in (1.3). The initial data (2.10) are inserted and integrated. The derivative with respect to x at the point $x = 0$ is then obtained as

$$u_x(0,t) = \sum_{j=0}^{k} C_j (c_j^r - c_j^l (-1)^j) t^{\frac{j-1}{2}}. \tag{2.11}$$

A singularity occurs at time $t = 0$, if there is a jump of the left and right-hand side polynomial. But the function (2.11) can be integrated improperly over the time interval $[0, \Delta t]$ to give

$$\int_0^{\Delta t} u_x(0,t)\,\mathrm{d}t = \sum_{j=0}^{k} C_j (c_j^r - c_j^l (-1)^j) \frac{2}{1+j} \Delta t^{\frac{j+1}{2}}. \tag{2.12}$$

with the recursive definition of the coefficients C_j:

$$C_j = 2j\kappa C_{j-2} \quad \text{with} \quad C_0 = \frac{1}{2\sqrt{\pi\kappa}} \quad \text{and} \quad C_1 = \frac{1}{2}. \tag{2.13}$$

The Eq. (2.12) gives the exact value for the time integrated diffusive flux of the generalized Riemann problem at the point $x = 0$.

Using the abbreviations

$$\begin{aligned} u^{\pm}_{i+\frac{1}{2}} &:= \lim_{\varepsilon \mapsto 0} u\left(x_{i+\frac{1}{2}} \pm \varepsilon\right), \\ [\![u]\!]_{i+\frac{1}{2}} &:= u^{+}_{i+\frac{1}{2}} - u^{-}_{i+\frac{1}{2}}, \\ \{u\}_{i+\frac{1}{2}} &:= \frac{1}{2}\left(u^{+}_{i+\frac{1}{2}} + u^{-}_{i+\frac{1}{2}}\right), \end{aligned} \tag{2.14}$$

the flux can be written for any k in the form

$$\int_0^{\Delta t} \kappa u_x\left(x_{i+\frac{1}{2}}, t\right) \mathrm{d}t := \sum_{j=0}^{\#\frac{k}{2}} \left(\eta \widetilde{C}_{2j} \left[\!\!\left[\frac{\partial^{2j} u}{\partial x^{2j}} \left(x_{i+\frac{1}{2}}, 0\right) \right]\!\!\right] + \left\{ \frac{\partial^{2j} u_x}{\partial x^{2j}} \left(x_{i+\frac{1}{2}}, 0\right) \right\} \right) \frac{(\kappa \Delta t)^{j+1}}{(j+1)!} \tag{2.15}$$

with the coefficients $\widetilde{C}_{2j} = 4^j \begin{pmatrix} 2j+1 \\ j \end{pmatrix}^{-1}$, $\eta := \frac{1}{\sqrt{\pi\kappa\Delta t}}$ and the integer value

$$\#\frac{k}{2} := \begin{cases} \frac{k}{2} & \text{for } k \text{ even}, \\ \frac{k+1}{2} & \text{for } k \text{ odd} \end{cases} \tag{2.16}$$

The formulation in this form shows that the dGRP flux consists of two parts. One part contains the jump of even order derivatives or for $j = 0$ the jump of the polynomial values, while the second part contains the arithmetic mean of odd derivatives.

2.2. Experimental order of convergence

In Table 1, we show the experimental order of convergence obtained for the diffusion problem (1.2) with the sinusoidal initial data

$$q(x) = \sin(2\pi x), \quad x \in [0,1], \tag{2.17}$$

for $t_{\text{end}} = 0.1$. A similar problem was investigated for discontinuous Galerkin schemes by Zhang and Shu in [27]. For the second order TVD reconstruction we obtain a first order accurate method for the diffusion equation. If the order of the reconstruction is increased to three using the WENO approach, then we obtain a second order method. This behavior is continued for higher orders and is shown in Table 1. The decrease of the order by one is due to the fact that the solution u is indeed reconstructed with $(k+1)$-th order of accuracy, but

Table 1
Experimental order of convergence for the FV scheme with the dGRP flux

#Cells	L_2	$\mathcal{O}_{L_2}$	L_1	$\mathcal{O}_{L_1}$
2nd Order TVD				
8	5.5655235E−03		5.0311993E−03	
16	2.3039823E−03	1.3	2.0691503E−03	1.3
32	1.0929452E−03	1.1	9.8316301E−04	1.1
64	5.3708225E−04	1.0	4.8342527E−04	1.0
3rd Order WENO				
8	1.1703535E−03		1.0479232E−03	
16	2.8839127E−04	2.0	2.5929233E−04	2.0
32	7.0252461E−05	2.0	6.3231695E−05	2.0
64	1.7525155E−05	2.0	1.5777174E−05	2.0
4th Order WENO				
8	5.2375353E−04		4.6729867E−04	
16	5.8686240E−05	3.2	5.2774386E−05	3.1
32	6.9615193E−06	3.1	6.2662748E−06	3.1
64	8.6228409E−07	3.0	7.7629596E−07	3.0
7th Order WENO				
8	1.1142303E−05		1.0040673E−05	
16	1.4126312E−07	6.3	1.2663577E−07	6.3
32	1.9654437E−09	6.2	1.7685081E−09	6.2
64	2.9905016E−11	6.0	2.6922175E−11	6.0

in the flux calculation we need the first spatial derivative which is then k-th order accurate only. For all calculations we chose a time step as given by $\Delta t = \Delta x^2/2\kappa$ which corresponds to the usual stability constraint for explicit schemes. The optimal accuracy is obtained with this choice. The accuracy decreases, if smaller time steps are chosen.

3. Discontinuous Galerkin for diffusion

3.1. Basic considerations

Bassi and Rebay [4] considered in 1997 discontinuous approximations of diffusion terms for the compressible Navier–Stokes equations. Cockburn and Shu [10] proposed in 1998 the so-called local discontinuous Galerkin (LDG) methods by generalizing this approach, for further historical remarks see Arnold et al. [3]. Bassi and Rebay as well as Cockburn and Shu started from the reformulation of the diffusion equation into a system of first order equations by introducing an additional variable q:

$$\begin{aligned} u_t - \kappa q_x &= 0, \\ q - u_x &= 0. \end{aligned} \tag{3.18}$$

As usual, these equations are multiplied by test functions and integrated over an arbitrary grid cell. Partial integration introduces fluxes at the boundary points of the grid cell in both equations. In the first equation, the flux is determined by $q = u_x$, while in the second one the flux determined by the function u itself.

We start in the following from the diffusion equation in the usual formulation (1.2) and apply partial integration twice. First, we multiply this equation with a test function and integrate over the grid cell Q_i:

$$\int_{Q_i} (u_t - \kappa u_{xx})\varphi \, \mathrm{d}x = 0. \tag{3.19}$$

We then use a first partial integration to get

$$\int_{Q_i} u_t \varphi \, \mathrm{d}x - [\kappa u_x \varphi]_{\partial Q_i} + \int_{Q_i} \kappa u_x \varphi_x \, \mathrm{d}x = 0, \tag{3.20}$$

where ∂Q_i denotes the two boundary points of Q_i. A second partial integration results in the following weak formulation of the diffusion equation:

$$\int_{Q_i} u_t \varphi \, \mathrm{d}x - [\kappa u_x \varphi]_{\partial Q_i} + [\kappa u \varphi_x]_{\partial Q_i} - \int_{Q_i} \kappa u \varphi_{xx} \, \mathrm{d}x = 0. \tag{3.21}$$

Also in this formulation two fluxes at the grid cell interface occur, one determined by u_x and the other by u.

We next introduce the approximation $u_h \in V_h$ of the solution u with

$$V_h = \{v : v \text{ is a polynomial of degree at most } k \text{ for } x \in Q_i,\ i = 1, \ldots, N\}. \tag{3.22}$$

The approximation may jump at a grid cell interface. In the semi-discrete setting as considered up to now the coefficients of the polynomials, i.e. the degrees of freedom, are still functions in time. We also replace the test functions by piecewise polynomials, but do not change their notation for simplicity.

The DG scheme is completely defined, if appropriate numerical fluxes for u_h and $(u_h)_x$ are specified. We assume that these fluxes cover the exchange across the boundaries well and that the integral is determined by interior values only. We can then reformulate (3.21) by using the identity

$$\int_{Q_i} u_h \varphi_{xx} \, \mathrm{d}x = [u_h \varphi_x]_{\mathrm{int}} - \int_{Q_i} (u_h)_x \varphi_x \, \mathrm{d}x \tag{3.23}$$

with the interior values

$$[u_h \varphi_x]_{\mathrm{int}} := (u_h)^-_{i+\frac{1}{2}} (\varphi_x)^-_{i+\frac{1}{2}} - (u_h)^+_{i-\frac{1}{2}} (\varphi_x)^+_{i-\frac{1}{2}}, \tag{3.24}$$

to evaluate the integral in Eq. (3.21). Inserting this identity into (3.21) leads to

$$\int_{Q_i} (u_h)_t \varphi \, \mathrm{d}x - [\kappa (u_h)_x \varphi]_{\partial Q_i} + [\kappa u_h \varphi_x]_{\partial Q_i} - [\kappa u_h \varphi_x]_{\mathrm{int}} + \int_{Q_i} \kappa (u_h)_x \varphi_x \, \mathrm{d}x = 0 \tag{3.25}$$

as a general weak formulation for the DG method. The formulation (3.25) directly corresponds to the general formulation based on the system (3.18).

Adding up Eq. (3.25) over all grid cells leads to the general primal formulation of Arnold et al. in [2,3]. They showed that choosing numerical fluxes for u_h and $(u_h)_x$ will bring up a wide range of commonly used DG methods for diffusion equations.

3.2. DG scheme with dGRP flux for scalar linear diffusion

To use the dGRP flux with the DG approach we start from formulation (3.21). For the definition of the numerical fluxes for $(u_h)_x$ we consider the local dGRP problem at the grid cell interface and use the diffusion flux (2.15). In addition, we need the flux of u_h which is also obtained from the exact dGRP solution in the form

$$\int_0^{\Delta t} u(x_{i+\frac{1}{2}}, t) \, \mathrm{d}t := \left\{ u\left(x_{i+\frac{1}{2}}, 0\right) \right\} \Delta t + \sum_{j=1}^{\frac{\#^{k+1}}{2}} \left(\eta \widetilde{C}_{2j} \left[\!\!\left[\frac{\partial^{2(j-1)} u_x}{\partial x^{2(j-1)}} \left(x_{i+\frac{1}{2}}, 0\right) \right]\!\!\right] + \left\{ \frac{\partial^{2j} u}{\partial x^{2j}} \left(x_{i+\frac{1}{2}}, 0\right) \right\} \right) \frac{\kappa^j \Delta t^{j+1}}{(j+1)!}. \tag{3.26}$$

We note that the flux for u_h and for $(u_h)_x$ are *conservative*, in the sense that they are uniquely defined for every grid cell interface. Arnold et al. [3] showed that this property is related to the *adjoint consistence* property of the corresponding primal formulation, which guarantees the optimal order for L_2 convergence. To demonstrate the accuracy of the dGRP-DG scheme, convergence tests for the unsteady linear diffusion Eq. (1.2) with sinusoidal initial data (2.17) and $t_{\mathrm{end}} = 0.1$ have been performed and are shown in Table 2. We observe that for polynomials with degree k we get the optimal convergence rate of $k + 1$ as expected.

Table 2
Experimental order of convergence of the DG method based on the dGRP flux

#Cells	L_2	$\mathcal{O}_{L_2}$	L_1	$\mathcal{O}_{L_1}$
DG(k = 1)				
8	1.57E−03		1.395E−03	
16	4.10E−04	1.9	3.63E−04	1.9
32	1.04E−04	2.0	9.18E−05	2.0
64	2.60E−05	2.0	2.30E−05	2.0
DG(k = 2)				
8	3.01E−05		2.48E−05	
16	3.46E−06	3.1	2.77E−06	3.2
32	4.23E−07	3.0	3.32E−07	3.1
64	5.26E−08	3.0	4.08E−08	3.0
DG(k = 3)				
4	2.02E−05		1.79E−05	
8	1.32E−06	3.9	1.05E−06	4.1
16	8.40E−08	4.0	6.44E−08	4.0
32	5.28E−09	4.0	4.01E−09	4.0
DG(k = 4)				
4	1.49E−06		1.32E−06	
6	1.95E−07	5.0	1.49E−07	5.4
8	4.62E−08	5.0	3.78E−08	4.8
16	1.44E−09	5.0	1.15E−09	5.0
DG(k = 5)				
2	7.69E−06		6.64E−06	
4	9.60E−08	6.3	8.28E−08	6.3
6	8.69E−09	5.9	7.08E−09	6.1
8	1.57E−09	6.0	1.25E−09	6.0

In these calculations, the time step Δt was always chosen as large as possible. For an explicit DG scheme for diffusion it is determined by a parabolic stability requirement in the form

$$\kappa \Delta t = \frac{\Delta x^2}{d^2}. \tag{3.27}$$

We remind that for the simplest second order explicit finite difference scheme we have $d = \sqrt{2}$ in one space dimension. The stability requirement for a DG scheme depends on the order of accuracy and becomes more stringent for higher orders. Numerical experiments gave us for d the minimal values 1.796, 5.000, 8.165, 16.43, 24.6 for the orders 2, 3, 4, 5 and 6, respectively. Combining (3.27) with η in the flux definitions (2.15) and (3.26) leads to

$$\eta = \frac{1}{\sqrt{\pi \kappa \Delta t}} = \frac{1}{\sqrt{\pi}} \frac{d}{\Delta x}, \tag{3.28}$$

which relates η to the stability constraint and the space increment.

3.3. A comparison for P^1 elements

As the proposed DG scheme is an approximation simultaneously in space and time, the comparison with other DG scheme is difficult, as this scheme provides rather a semi-discrete approximation, whereas in other schemes, the time approximation is usually done in an ODE approach using a Runge–Kutta scheme. But in the simplest case, piecewise linear trial and test functions and the explicit Euler time approximation, a comparison becomes feasible and will be presented in the following. We thus assume a piecewise representation of the solution, linear in every grid cell, usually denoted by P^1. To abbreviate the notations we skip the index h, and simply write u for the approximate piecewise linear solution.

We start with the simplest but inconsistent possibility: the arithmetic mean of the first order derivative from the right and the left-hand side. The corresponding numerical fluxes read as

$$u|_{i\pm\frac{1}{2}} := u^{\mp}_{i\pm\frac{1}{2}} \quad \text{and} \quad u_x|_{i\pm\frac{1}{2}} := \{u_x\}_{i\pm\frac{1}{2}}. \tag{3.29}$$

This scheme is inconsistent, since it does not take into account the jumps at the grid cell interfaces, see, e.g. [27] for further details. The fully discrete form of the P^1 scheme using the arithmetic mean flux and the Euler time approximation reads as

$$\int_{Q_i} u|_{t_n}^{t_{n+1}} \varphi \,\mathrm{d}x + \Delta t \int_{Q_i} \kappa u_x^n \varphi_x \,\mathrm{d}x - \kappa \Delta t \left(\{u_x^n\}_{i+\frac{1}{2}} \varphi^-_{i+\frac{1}{2}} - \{u_x^n\}_{i-\frac{1}{2}} \varphi^+_{i-\frac{1}{2}} \right) = 0. \tag{3.30}$$

An idea of the FE community to get consistency was to add an additional term that penalizes the jumps. The first penalty scheme was already proposed by Nitsche [20] for elliptic equations within the primal formulation, called the *Interior Penalty* or *Symmetric Interior Penalty (SIP)* scheme. It was studied in detail, e.g. by Arnold [1]. For the fully discrete flux formulation we can use the formulation (3.25) with the numerical fluxes

$$u|_{i\pm\frac{1}{2}} := \{u\}_{i\pm\frac{1}{2}} \quad \text{and} \quad u_x|_{i\pm\frac{1}{2}} := \{u_x\}_{i\pm\frac{1}{2}} + \bar{\eta} [\![u]\!]_{i\pm\frac{1}{2}}, \tag{3.31}$$

and the Euler time integration to get

$$\begin{aligned} &\int_{Q_i} u|_{t_n}^{t_{n+1}} \varphi \,\mathrm{d}x + \Delta t \int_{Q_i} \kappa u_x^n \varphi_x \,\mathrm{d}x - \kappa \Delta t \left(\{u_x^n\}_{i+\frac{1}{2}} \varphi^-_{i+\frac{1}{2}} - \{u_x^n\}_{i-\frac{1}{2}} \varphi^+_{i-\frac{1}{2}} \right) \\ &\quad + \frac{1}{2} \kappa \Delta t \left([\![u^n]\!]_{i+\frac{1}{2}} (\varphi_x)^-_{i+\frac{1}{2}} + [\![u^n]\!]_{i-\frac{1}{2}} (\varphi_x)^+_{i-\frac{1}{2}} \right) - \kappa \Delta t \bar{\eta} \left([\![u^n]\!]_{i+\frac{1}{2}} \varphi^-_{i+\frac{1}{2}} - [\![u^n]\!]_{i-\frac{1}{2}} \varphi^+_{i-\frac{1}{2}} \right) = 0. \end{aligned} \tag{3.32}$$

For this, Houston, Schwab and Süli showed in [16] that the penalization parameter has to be chosen for stability reasons as

$$\bar{\eta} = C \frac{k^2}{\Delta x}, \tag{3.33}$$

where k denotes again the degree of the polynomial and C is a sufficiently large constant.

In the following, we consider the dGRP approach. The starting point of the scheme is formulation (3.25). The numerical fluxes are given by (2.15) and (3.26) for $k = 1$ and read for the semi-discrete case as

$$\begin{aligned} u\left(x_{i+\frac{1}{2}}, t + t_n\right) &:= \{u^n\}_{i+\frac{1}{2}} + \sqrt{\frac{\kappa t}{\pi}} [\![u_x^n]\!]_{i+\frac{1}{2}}, \\ u_x\left(x_{i+\frac{1}{2}}, t + t_n\right) &:= \{u_x^n\}_{i+\frac{1}{2}} + \frac{1}{2\sqrt{\pi \kappa t}} [\![u^n]\!]_{i+\frac{1}{2}}. \end{aligned} \tag{3.34}$$

The fully-discrete fluxes are obtained by integration in time. Inserting this in (3.25) leads to the following DG scheme:

$$\begin{aligned} &\int_{Q_i} u|_{t_n}^{t_{n+1}} \varphi \,\mathrm{d}x + \Delta t \int_{Q_i} \kappa u_x^n \varphi_x \,\mathrm{d}x - \kappa \Delta t \left(\{u_x^n\}_{i+\frac{1}{2}} \varphi^-_{i+\frac{1}{2}} - \{u_x^n\}_{i-\frac{1}{2}} \varphi^+_{i-\frac{1}{2}} \right) \\ &\quad + \frac{1}{2} \kappa \Delta t \left([\![u^n]\!]_{i+\frac{1}{2}} (\varphi_x)^-_{i+\frac{1}{2}} + [\![u^n]\!]_{i-\frac{1}{2}} (\varphi_x)^+_{i-\frac{1}{2}} \right) - \kappa \Delta t \eta \left([\![u^n]\!]_{i+\frac{1}{2}} \varphi^-_{i+\frac{1}{2}} - [\![u^n]\!]_{i-\frac{1}{2}} \varphi^+_{i-\frac{1}{2}} \right) \\ &\quad + \frac{2}{3} (\kappa \Delta t)^2 \eta \left([\![u_x^n]\!]_{i+\frac{1}{2}} (\varphi_x)^-_{i+\frac{1}{2}} - [\![u_x^n]\!]_{i-\frac{1}{2}} (\varphi_x)^+_{i-\frac{1}{2}} \right) = 0. \end{aligned} \tag{3.35}$$

The comparison with the SIP scheme (3.32) indicates that the first two flux terms are identical. The third one only differs by the 'penalization' constant. If the penalization parameter $\bar{\eta}$ in the interior penalty scheme is defined to be η from (3.28), then also the third term of the flux is identical. In numerical experiments, we obtained stability and the constraint

$$\frac{\kappa \Delta t}{\Delta x^2} \leqslant 0.16, \tag{3.36}$$

with the minimal penalty constant

$$\bar{\eta}_{\min} = \frac{1}{\sqrt{\pi\kappa\Delta t_{\max}}} \approx \frac{1.41}{\Delta x}. \tag{3.37}$$

The additional fourth flux term of the dGRP flux has no counterpart in the SIP-DG scheme. In our numerical studies we observed, that this term increases the stability and allows larger time steps for the P^1 case, see (3.37) and (3.38). For higher order approximations it turned out, that the effect of these terms decrease. The P^1 term corresponds to one found by van Leer and Nomura [25] who constructed a DG scheme for diffusion based on a smooth recovery at the grid cell interfaces. They state that the class of interior penalty schemes should be extended to include this additional term. Instead of (3.37) the minimal penalty constant of the dGRP-DG scheme is given by

$$\eta_{\min} \approx \frac{1}{\Delta x}. \tag{3.38}$$

We remark that the dGRP approach is an approach simultaneously in space and time, for which the semi-discrete version depends explicitly on time t, see (3.34). Thus, for pure elliptic problems the dGRP-DG scheme cannot directly be used.

4. Diffusion systems

4.1. Equations and basic formulation

In this section, we extend the dGRP-DG scheme to solve diffusion problems of the form

$$U_t - F(U, U_x)_x = U_t - (\underline{D}(U)U_x)_x = 0, \tag{4.39}$$

with the vector of unknowns $U := (u_1, \ldots, u_M)^{\mathrm{T}} \in \mathbb{R}^M$, and the positive semi-definite diffusion matrix $\underline{D}(U) \in \mathbb{R}^{M\times M}$. We consider this extension within the DG framework, but the application to FV schemes can be done in an analogous way.

The base of our approximation is a weak formulation, which is derived again with two integration by parts. We multiply Eq. (4.39) with a test function $\Phi = (\phi_1, \ldots, \phi_M)$ and integrate over an arbitrary grid cell Q_i:

$$\int_{Q_i} (U_t - F(U, U_x)_x) \cdot \Phi(x)\,\mathrm{d}x = 0, \tag{4.40}$$

where "$\cdot$" denotes the scalar product in $\mathbb{R}^M$. We proceed with the first integration by parts

$$\int_{Q_i} U_t \cdot \Phi(x)\,\mathrm{d}x - [F(U, U_x) \cdot \Phi]_{\partial Q_i} + \int_{Q_i} F(U, U_x) \cdot \Phi_x(x)\,\mathrm{d}x = 0. \tag{4.41}$$

Due to the second order character of the problem (4.39), we are able to perform another integration by parts of the volume integral. We first reformulate this term using the homogeneity property of the viscous flux with respect to the gradient

$$\int_{Q_i} F(U, U_x) \cdot \Phi_x(x)\,\mathrm{d}x = \int_{Q_i} (\underline{D}(U)U_x) \cdot \Phi_x(x)\,\mathrm{d}x = \int_{Q_i} U_x \cdot \left(\underline{D}(U)^{\mathrm{T}}\Phi_x(x)\right)\mathrm{d}x, \tag{4.42}$$

and proceed with the second integration by parts:

$$\int_{Q_i} U_x \cdot \underline{D}(U)^{\mathrm{T}}\Phi_x(x)\,\mathrm{d}x = [U \cdot \underline{D}(U)^{\mathrm{T}}\Phi_x]_{\partial Q_i} - \int_{Q_i} U \cdot \frac{\partial}{\partial x}(\underline{D}(U)^{\mathrm{T}}\Phi_x(x))\,\mathrm{d}x. \tag{4.43}$$

For our DG discretization we introduce the approximation $U_h = (u_1^h, \ldots, u_M^h)^{\mathrm{T}}$ of the solution U. Again, the considered approximation is for each component a polynomial of degree at most k in every grid cell. We choose orthogonal trial and test functions, namely the normalized Legendre polynomials, and use the notation

$$U_i^n := U_h(x,t_n) = \sum_{p=1}^{k+1} \widehat{U}_{i,p}^n \phi_p(x), \quad \text{for } x \in Q_i. \tag{4.44}$$

To easily evaluate the *new* viscous volume integral, we use the following identity

$$-\int_{Q_i} U_h \cdot \frac{\partial}{\partial x}(\underline{D}(U_h)^{\mathrm{T}} \Phi_x(x))\,\mathrm{d}x = -[U_h \cdot \underline{D}(U_h)^{\mathrm{T}} \Phi_x]_{\mathrm{int}} + \int_{Q_i} (U_h)_x \cdot \underline{D}(U_h)^{\mathrm{T}} \Phi_x(x)\,\mathrm{d}x. \tag{4.45}$$

Combining these derivations yield the weak formulation of the diffusion system (4.39) as

$$\begin{aligned}&\int_{Q_i} (U_h)_t \cdot \Phi(x)\,\mathrm{d}x - [F(U_h,(U_h)_x)\cdot\Phi]_{\partial Q_i} + [U_h \cdot \underline{D}(U_h)^{\mathrm{T}}\Phi_x]_{\partial Q_i} - [U_h \cdot \underline{D}(U_h)^{\mathrm{T}}\Phi_x]_{\mathrm{int}} \\ &\quad + \int_{Q_i} F(U_h,(U_h)_x)\cdot\Phi_x(x)\,\mathrm{d}x = 0.\end{aligned} \tag{4.46}$$

The back and forth integration by parts generates two additional flux terms in (4.46) compared to formulation (4.41). For a smooth solution both terms are equal and cancel out each other, while for a discontinuous approximation a term depending on the left and right value of the approximate solution is generated. A similar trick is used by Bassi and Rebay [5], in the context of a mixed formulation to introduce the so-called global lifting operator.

With the identity

$$U_h \cdot (\underline{D}(U_h)^{\mathrm{T}}\Phi_x) = (\underline{D}(U_h)U_h)\cdot\Phi_x = F(U_h,U_h)\cdot\Phi_x, \tag{4.47}$$

we can rewrite the semi-discrete weak formulation (4.46) in terms of the viscous flux F as

$$\begin{aligned}&\int_{Q_i} (U_h)_t \cdot \Phi\,\mathrm{d}x - [F(U_h,(U_h)_x)\cdot\Phi]_{\partial Q_i} + \int_{Q_i} F(U_h,(U_h)_x)\cdot\Phi_x\,\mathrm{d}x + [F(U_h,U_h)\cdot\Phi_x]_{\partial Q_i} \\ &\quad - [F(U_h,U_h)\cdot\Phi_x]_{\mathrm{int}} = 0,\end{aligned} \tag{4.48}$$

which has some advantages for implementation issues. To keep notations short we will use in the following U instead of U_h.

4.2. The linear case

We first extend the dGRP flux to the case of a linear system, where $\underline{D}$ is a constant matrix, with $F = F(U_x) = \underline{D}U_x$. Analogous to the scalar case, we need the time integrals of the viscous fluxes. We restrict ourselves to the case where $\underline{D}$ has a full set of eigenvectors V_p. This allows to construct the dGRP solution in the spirit of hyperbolic Riemann solvers. We start with the initial value problem for the diffusion system

$$\begin{aligned}&U_t - \underline{D}U_{xx} = 0,\\ &U(x,0) = \begin{cases} U_r(x) & \text{for } x > 0,\\ U_l(x) & \text{for } x < 0,\end{cases}\end{aligned} \tag{4.49}$$

where U_r and U_l are polynomials. With the above mentioned assumption it is then possible to introduce the new variables

$$W := \underline{V}^{-1}U, \tag{4.50}$$

where the columns of $\underline{V}$ are the eigenvectors V_p. Inserting this into Eq. (4.49) leads to an uncoupled system of the form

$$\begin{aligned}&(w_p)_t - \lambda_p (w_p)_{xx} = 0,\\ &w_p(x,0) = \begin{cases} (w_p)_r(x) & \text{for } x > 0,\\ (w_p)_l(x) & \text{for } x < 0,\end{cases}\\ &p = 1,\ldots,M,\end{aligned} \tag{4.51}$$

where λ_p denotes the corresponding eigenvalue of the diffusion matrix. For each scalar equation we can use the exact dGRP formulae (2.15) and (3.26). After re-substituting (4.50) we obtain the exact dGRP solution, which completes the dGRP-DG scheme for linear diffusion systems.

For the P^1 case this reads as

$$\begin{aligned}\int_0^{\Delta t} U_x\left(x_{i+\frac{1}{2}},t\right)\mathrm{d}t &= \left(\left\{U_x\left(x_{i+\frac{1}{2}},0\right)\right\}+\frac{d(k)}{\sqrt{\pi}\Delta x}\underline{P}\left[\!\left[U(x_{i+\frac{1}{2}},0)\right]\!\right]\right)\Delta t,\\ \int_0^{\Delta t} U\left(x_{i+\frac{1}{2}},t\right)\mathrm{d}t &= \left(\left\{U\left(x_{i+\frac{1}{2}},0\right)\right\}+\frac{2\Delta x}{3\sqrt{\pi}d(k)}\underline{Q}\left[\!\left[U_x\left(x_{i+\frac{1}{2}},0\right)\right]\!\right]\right)\Delta t,\end{aligned} \tag{4.52}$$

with the matrices

$$\underline{P} := \underline{V\Lambda V}^{-1} \quad \text{and} \quad \underline{Q} := \underline{V\Theta V}^{-1}, \tag{4.53}$$

where $\underline{\Lambda}$ and $\underline{\Theta}$ are diagonal matrices with entries $\underline{\Lambda}[p,p]=\sqrt{\frac{\lambda_{\max}}{\lambda_p}}$ and $\underline{\Theta}[p,p]=\sqrt{\frac{\lambda_p}{\lambda_{\max}}}$. The time step is calculated as $\Delta t=\frac{\Delta x^2}{d(k)^2\lambda_{\max}}$, with $\lambda_{\max}=\max_{p=1,\ldots,M}\lambda_p$ and the stability numbers $d(k)$ from the scalar case.

4.3. The nonlinear case

For a general nonlinear system (4.39) no exact solution of the dGRP is available. Hence, we first approximate the integrals by an appropriate high order quadrature formulae. For a trial function of degree k, we choose $J := k+1$ Gauss points ξ_j in space and $L := \#\frac{k+4}{4}$ Gauss points τ_l in time. Concerning the number of Gauss points in time we take into account the parabolic time step restriction $\Delta t \sim \Delta x^2$. We only have to integrate exactly a time polynomial of degree $\#\frac{k+1}{2}$, as this still guarantees the high order with respect to the discretization parameter Δx. The corresponding Gauss weights in space and time are denoted by $\widetilde{\omega}_j$ and ω_l, respectively. The fully discrete DG scheme is then given by

$$\begin{aligned}\widehat{U}_{i,p}^{n+1}-\widehat{U}_{i,p}^{n} = &-\sum_{l=1}^{L(k)}\omega_l\left([G(\tau_l)\phi_{i,p}]_{\partial Q_i}+\left[H^*(\tau_l)\frac{\partial\phi_{i,p}}{\partial x}\right]_{\text{int}}-\left[H(\tau_l)\frac{\partial\phi_{i,p}}{\partial x}\right]_{\partial Q_i}\right)\\ &+\sum_{l=1}^{L(k)}\sum_{j=1}^{J(k)}\widetilde{\omega}_j\omega_l F(U_h(\xi_j,\tau_l),(U_h)_x(\xi_j,\tau_l))\frac{\partial\phi_{i,p}}{\partial x}(\xi_j),\quad p=1,\ldots,M,\end{aligned} \tag{4.54}$$

where G, H^* and H denote the different numerical fluxes.

To evaluate the right-hand side of this Eq. (4.54) we need values of the approximation U_h at all the space–time Gauss points. These values are obtained from the space–time Taylor expansion

$$U_h(\xi,\tau) := U_i^n(x_i)+\sum_{j=1}^{k}\frac{1}{j!}\left(\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}\right)^j U_i^n(x_i),\quad \text{for } (\xi,\tau)\in Q_i\times[t_n;t_n+\Delta t), \tag{4.55}$$

where x_i denotes the barycenter of the grid cell. This follows the evolution operator based time integration, proposed by Harten et al. [15] already in 1987, in the framework of finite volume schemes for the Euler equations. For a piecewise linear approximation this approach equals to van Leer's well-known MUSCL approach [7]. While the space derivatives are readily available from the polynomial approximation at time t_n the time and the mixed space–time derivatives have to be calculated using the so-called Cauchy–Kovalevskaya (CK) procedure. In this procedure, the time derivatives are computed from space derivatives using successively the governing equations. Within the construction of numerical methods it was used first in the Lax–Wendroff method [17] to get second order accuracy in space and time. Later on, Toro and his co-workers proposed in [23,24] for advection equations the class of finite volume ADER schemes which has as main building block the approximation of the GRP based on the CK procedure. We also refer to Dumbser and Munz [12] and Dumbser [11] who introduced an efficient way to implement the CK procedure for nonlinear systems with the aid of the generalized Leibniz rule. Based on the CK procedure a generalization of the Lax–Wendroff scheme to arbitrary order and multi dimensions is given in [19].

Using the space–time approximation (4.55) it is straightforward to calculate the volume integrals in the weak formulation. The values of the integrand at the space–time Gauss points are given by U_h evaluated at these points. The arguments for the numerical fluxes between the grid cells are also given by the space–time Taylor expansions from the left and the right cell.

To conclude the derivation of the scheme, we have to define appropriate numerical fluxes across the grid cell interfaces. We first linearize the nonlinear fluxes at each time Gauss point about the arithmetic mean $\{U\}_{i+\frac{1}{2}}$:

$$G_{i+\frac{1}{2}}(\tau_l) = D_{i+\frac{1}{2},l} U_x\left(x_{i+\frac{1}{2}}, \tau_l\right) := D\left(\{U\}_{i+\frac{1}{2}}(\tau_l)\right) U_x\left(x_{i+\frac{1}{2}}, \tau_l\right). \tag{4.56}$$

This seems to be a natural choice, as diffusion does not prefer any direction. According to Section 4.2, we choose the following structure for our gradient approximation

$$U_x\left(x_{i+\frac{1}{2}}, \tau_l\right) = \left\{U_x(x_{i+\frac{1}{2}}, \tau_l)\right\} + \frac{d(k)}{\sqrt{\pi}\Delta x} \underline{P}_{i+\frac{1}{2},l} \left[\!\left[U\left(x_{i+\frac{1}{2}}, \tau_l\right)\right]\!\right], \tag{4.57}$$

which yields the following numerical flux

$$G_{i+\frac{1}{2}}(\tau_l) = F\left(\{U\}_{i+\frac{1}{2},l}, \{U_x\}_{i+\frac{1}{2},l} + \frac{d(k)}{\sqrt{\pi}\Delta x} \underline{P}_{i+\frac{1}{2},l} [\![U]\!]_{i+\frac{1}{2},l}\right). \tag{4.58}$$

For the second numerical flux $H_{i+\frac{1}{2}}(\tau_l)$ we proceed in an analogous way and obtain

$$H_{i+\frac{1}{2}}(\tau_l) = F\left(\{U\}_{i+\frac{1}{2},l}, \{U\}_{i+\frac{1}{2},l} + \frac{2\Delta x}{3\sqrt{\pi} d(k)} \underline{Q}_{i+\frac{1}{2},l} [\![U_x]\!]_{i+\frac{1}{2},l}\right). \tag{4.59}$$

To enhance computational efficiency we propose a simplification of the internal flux by

$$H^*_{i+\frac{1}{2}}(\tau_l) := \underline{D}\left(U^-_{i+\frac{1}{2}}(\tau_l)\right) U^-_{i+\frac{1}{2}}(\tau_l) \approx \underline{D}_{i+\frac{1}{2},l} U^-_{i+\frac{1}{2}}(\tau_l), \tag{4.60}$$

which allows to combine the fluxes H^* and H to

$$-H^*_{i+\frac{1}{2}} + H_{i+\frac{1}{2}} \approx \frac{1}{2} F\left(\{U\}_{i+\frac{1}{2}}, [\![U]\!]_{i+\frac{1}{2}} + \frac{4\Delta x}{3\sqrt{\pi} d(k)} \underline{Q}_{i+\frac{1}{2},l} [\![U_x]\!]_{i+\frac{1}{2}}\right). \tag{4.61}$$

We observed in numerical experiments, that the last term involving the jump of the derivative $[\![U_x]\!]$ enhances the stability. However, analogous to the scalar case, for higher order approximations the effect is very weak, which may be due to the fact, that this term is multiplied with $\frac{4\Delta x}{3\sqrt{\pi} d(k)}$, which decreases for higher order. For computational efficiency we drop this term out and use

$$-H^*_{i+\frac{1}{2}} + H_{i+\frac{1}{2}} \approx \frac{1}{2} F\left(\{U\}_{i+\frac{1}{2}}, [\![U]\!]_{i+\frac{1}{2}}\right). \tag{4.62}$$

In our calculations in the next subsection we also modified the jump term of the numerical flux $G_{i+\frac{1}{2}}(\tau_l)$. As in the considered example the eigenvalues of the diffusion matrix do not differ much, we set $\underline{\Lambda}_{i+\frac{1}{2}}[\bar{p}, p] \approx 1$ to obtain

$$G_{i+\frac{1}{2}}(\tau_l) = F\left(\{U\}_{i+\frac{1}{2},l}, \{U_x\}_{i+\frac{1}{2},l} + \frac{d(k)}{\sqrt{\pi}\Delta x} [\![U]\!]_{i+\frac{1}{2},l}\right). \tag{4.63}$$

We note, that in the system case the numerical flux of the SIP-DG scheme in the form

$$G^{\mathrm{SIP}}_{i+\frac{1}{2}}(\tau_l) = \{F(U, U_x)\}_{i+\frac{1}{2},l} + C \frac{k^2}{\Delta x} \lambda_{\max} [\![U]\!]_{i+\frac{1}{2},l} \tag{4.64}$$

differs from the dGRP approach, as the jump term is not multiplied with the diffusion matrix. We observed in numerical experiments that for the compressible Navier–Stokes equation, a SIP-type jump term leads to a discretization with sub-optimal order of accuracy: For polynomial trial functions with even degree k we only get a convergence rate k. Numerical investigations indicate, that in the case of mixed hyperbolic/parabolic prob-

lems a jump term should be avoided in the hyperbolic part. If the jump term is multiplied with the diffusion matrix this is automatically guaranteed.

In the nonlinear case, we approximated the time integrals by Gauss quadrature and obtain the values at the Gauss points from a Taylor expansion. We conclude this derivation by applying this "nonlinear" procedure to the scalar linear case. With the above introduced approximations, the numerical flux reads in this case as

$$\int_0^{\Delta t} \kappa u_x\left(x_{i+\frac{1}{2}},t\right)\mathrm{d}t := \sum_{j=0}^{\frac{\#k}{2}}\left(\eta\left[\!\left[\frac{\partial^{2j}u}{\partial x^{2j}}\left(x_{i+\frac{1}{2}},0\right)\right]\!\right]+\left\{\frac{\partial^{2j}u_x}{\partial x^{2j}}\left(x_{i+\frac{1}{2}},0\right)\right\}\right)\frac{(\kappa\Delta t)^{j+1}}{(j+1)!}. \tag{4.65}$$

The comparison with the flux (2.15) obtained from the time integration of exact dGRP solution, indicates a small difference in the constants of higher derivatives. The values $\widetilde{C}_{2j}$ in (2.15) become slightly larger than one, while in (4.65) they are equal to one. If we consider smooth data, the jump cancels out and thus in this case the formulae (4.65) and (2.15) coincide.

4.4. Numerical results

To demonstrate the accuracy of the scheme and the experimental order of convergence we consider the following system of two equations

$$\begin{pmatrix}u_1\\u_2\end{pmatrix}_t=\left(\begin{pmatrix}u_2 & 0\\ u_2 & u_1\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}_x\right)_x+\begin{pmatrix}-0.2\mathrm{e}^{0.5x-0.2t}-0.75\mathrm{e}^{1.5x-1.2t}\\ -\mathrm{e}^{x-t}-2.25\mathrm{e}^{1.5x-1.2t}\end{pmatrix}, \tag{4.66}$$

for $(x,t)\in[0;1]\times[0;0.1]$. The exact solution is given by

Table 3
Experimental order of convergence

#Cells	$L_2(u_2)$	$\mathcal{O}_{L_2}$	$L_1(u_2)$	$\mathcal{O}_{L_1}$
DG(k = 1)				
8	1.20E−03		8.65E−04	
16	2.89E−04	2.1	2.11E−04	2.0
32	7.14E−05	2.0	5.26E−05	2.0
64	1.78E−05	2.0	1.31E−05	2.0
DG(k = 2)				
8	1.55E−05		1.02E−05	
16	1.99E−06	3.0	1.36E−06	2.9
32	2.53E−07	3.0	1.75E−07	3.0
64	3.16E−08	3.0	2.21E−08	3.0
DG(k = 3)				
4	4.98E−06		4.04E−06	
8	2.86E−07	4.1	2.30E−07	4.1
16	1.68E−08	4.1	1.30E−08	4.1
32	1.02E−09	4.0	7.64E−10	4.1
DG(k = 4)				
4	2.37E−08		1.70E−08	
6	3.16E−09	5.0	2.30E−09	4.9
8	7.57E−10	5.0	5.55E−10	4.9
16	2.41E−11	5.0	1.79E−11	5.0
DG(k = 5)				
2	4.46E−08		3.73E−08	
4	7.16E−10	6.0	6.07E−10	5.9
6	7.26E−11	5.6	6.12E−11	5.7
8	1.49E−12	5.5	1.26E−12	5.5

$$\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = \begin{pmatrix} e^{0.5x-0.2t} \\ e^{x-t} \end{pmatrix}, \tag{4.67}$$

if appropriate initial values and boundary values are described. For the numerical calculations the exact values at the boundary and at time $t = 0$ are prescribed.

In Table 3, the experimental order of convergence for the problem (4.66) with $t_{\text{end}} = 0.1$ and $d(k) = d_{\min}(k)$ are shown. We used for this calculations the numerical fluxes (4.61) and (4.62). The results clearly show that the desired order of accuracy is obtained.

5. Conclusions

We used the exact solution of the diffusive generalized Riemann problem to define a numerical flux for finite volume and discontinuous Galerkin schemes. The main advantage of this procedure is that the definition of the numerical flux is based on the same data as the advection flux, no different treatment of the diffusion terms is necessary, e.g. by assuming continuity at the grid cell interfaces. A difference to the hyperbolic case is that a scheme based on the Riemann problem with piecewise constant initial data leads to an inconsistent scheme. In the case of diffusion, it is necessary to take at least a piecewise linear approximation.

For finite volume schemes the order of convergence in space and time for smooth data are identical to the degree of the polynomials in the reconstruction. It has to satisfy the usual stability constraint for explicit finite difference schemes. If the dGRP flux is used within the discontinuous Galerkin framework the resulting scheme has the optimal order $k + 1$ simultaneously in space and time for an approximation with polynomials of degree k. A comparison with other discontinuous Galerkin schemes indicates, that the dGRP-DG scheme is strongly associated to the symmetric interior penalty (SIP-DG) scheme. It gives a physical foundation of penalization and a suggestion for a fix of the penalty constant in the SIP-DG scheme. The space–time character of the dGRP-DG scheme is reflected in the formula of the penalty constant where the time step Δt is included. The proposed scheme fits into an extended class of interior penalty schemes which also contains the recovery schemes of van Leer and Nomura [25].

We showed how to mimic the dGRP flux in an approximative way to nonlinear diffusion systems. A main building block of the presented approximation is a combination of a space–time expansion and the Cauchy–Kovalevskaya procedure. An alternative for time approximation would be the ODE approach using, e.g. a Runge Kutta time approximation.

The extension to advection diffusion equations can be obtained by adding a numerical advection flux at all time Gauss points. High order time consistency is given by the advection-diffusion CK procedure.

Acknowledgments

This research was supported by Deutsche Forschungsgemeinschaft (DFG). The authors thank Michael Dumbser and Tito Toro for many helpful discussions. We thank all the reviewers very much. We never got such extensive and constructive reviews. We think that the demands for revision strongly improved the paper.

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